Hydrodynamics from Anomaly Inflow

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Outline

- Topological phases of matter and anomaly inflow
- Bosonization in 1+1d and anomalies
- Hydrodynamic of perfect fluid in 3+1d
- Including 3+1d anomalies via anomaly inflow

Effective field theory, anomalies & hydrodynamics

- Topological phases of matter provide us new insights:

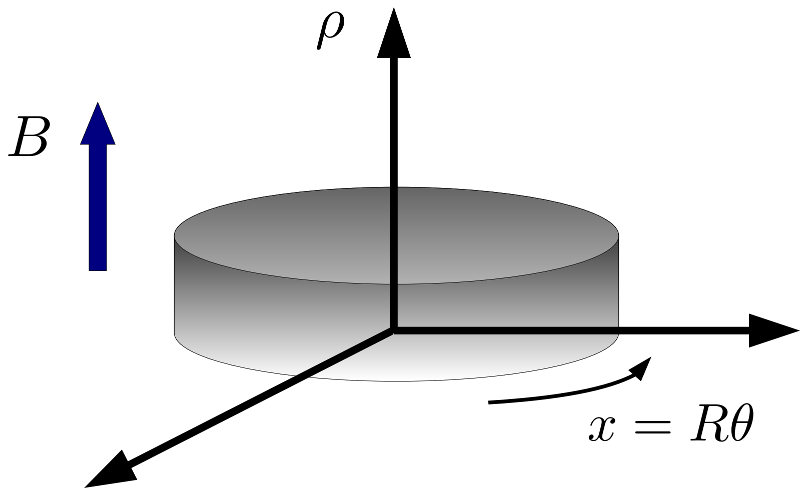
  - bulk gap and massless boundary excitations
  - both bosonic and fermionic field theory descriptions → bosonization
  - topological gauge theories & anomalies
  - well understood in 1+1d, e.g. QHE

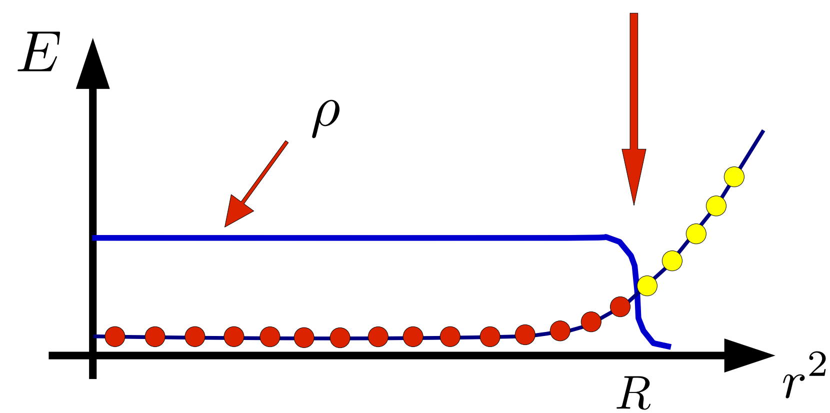
- Extension to 3+1d; we shall use hydrodynamics:

  - perfect barotropic fluid: T=0, s=const, dynamics from pressure & density
  - Lagrangian formulation
  - hydrodynamics with anomalies in 3+1d
  - bosonic `geometric' description of anomalies: universal response
  - many potential applications

Quantum Hall effect: bulk and edge

Filled Landau level: bulk gap, massless edge fermion





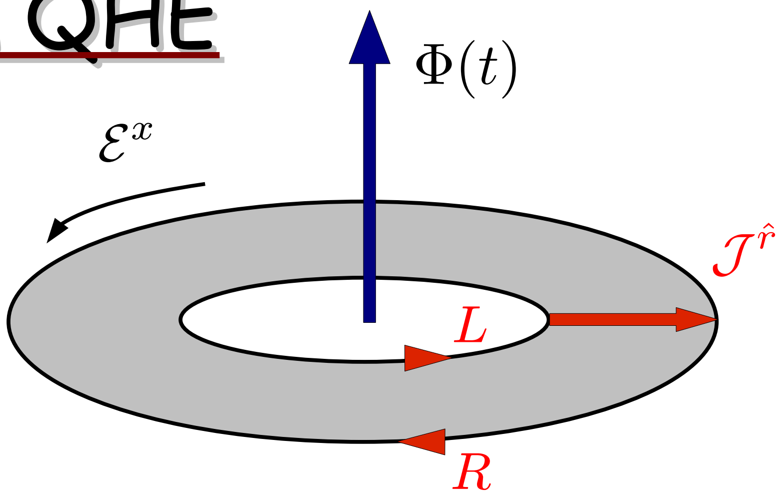
edge ~ Fermi surface: linearize energy $\varepsilon(k) = vk, \quad k = \frac{1}{R}(n - n_F), \quad n_F = N$

→ set $r = R = \ell\sqrt{N}$; massless chiral fermion in (1+1) dimensions $\psi(r, \theta, t)|_{r=R}$

→ fractional fillings $\nu = \frac{1}{3}, \frac{1}{5}, \ldots$ → interacting fermion → bosonization

scalar field $\theta(x - t)$ (c=1 chiral conformal field theory)

Anomaly inflow in QHE



- Add flux $\Phi(t)$; tangential electric field $\mathcal{E}^x$
- Hall current $\mathcal{J}^{\hat{r}} = \sigma_H \mathcal{E}^x$
- Hall current = chiral anomaly of 1+1d edge theory

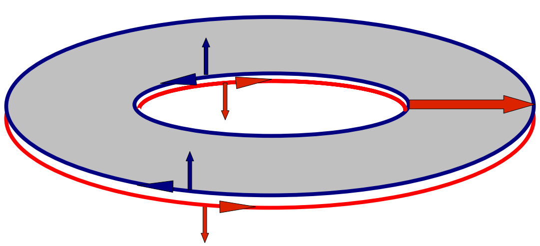
$$\oint dx\ \mathcal{J}^{\hat{r}} = \dot{Q}_{\text{edge}} = \oint dx\ \partial_\alpha J^\alpha, \qquad \partial_\alpha J^\alpha = -\frac{e}{2\pi}\varepsilon^{\beta\gamma}\partial_\beta A_\gamma, \quad \alpha = 0, 1,$$

- Hall current given by Chern-Simons topological theory in 2+1d

$$S_{CS}[A] = -\frac{1}{4\pi}\int d^3x\ AdA, \qquad A = A_\mu dx^\mu, \quad \mu = 0, 1, 2$$

$$\mathcal{J}^{\hat{r}} = \frac{\delta S_{CS}}{\delta A_{\hat{r}}} = -\frac{1}{2\pi}\varepsilon^{\hat{r}\alpha\beta}\partial_\alpha A_\beta = \partial_\alpha J^\alpha$$

- extends to non-chiral case (topological insulators)



→ **bulk+boundary system is gauge invariant**

1+1d bosonic theory and anomalies

$$S = \frac{1}{2}\int d^2x \; (\partial_\mu \theta)^2$$

- U(1) symmetry $\theta \to \theta + \mathrm{const.}$ but two conserved currents

$J^\mu = \partial_\mu \theta,$ $\quad \partial_\mu J^\mu = 0$ Noether current

$\tilde{J}^\mu = \varepsilon^{\mu\nu}\partial_\nu \theta,$ $\quad \partial_\mu \tilde{J}^\mu = 0$ topological (axial) current

- like Dirac fermion: $J^\mu = \bar{\psi}\gamma^\mu\psi,$ $\quad \tilde{J}^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$
- couple to corresponding backgrounds $A_\mu, \tilde{A}_\mu$

$$S = \int d^2x \frac{1}{2}\left(\partial_\mu\theta - A_\mu\right)^2 + \tilde{A}_\mu \varepsilon^{\mu\nu}\left(\partial_\nu \theta - A_\nu\right)$$

- gauge-invariant currents ("covariant currents") are anomalous

$J^\mu = \partial_\mu \theta - A_\mu,$ $\quad \partial_\mu J^\mu = -\varepsilon^{\mu\nu}\partial_\mu \tilde{A}_\nu,$ $\quad (e/2\pi \to 1)$

$\tilde{J}^\mu = \varepsilon^{\mu\nu}\left(\partial_\nu\theta - A_\nu\right),$ $\quad \partial_\mu \tilde{J}^\mu = -\varepsilon^{\mu\nu}\partial_\mu A_\nu$

- Anomalies reproduced at classical level in the bosonic effective theory

Checking the anomaly inflow

- Topological theory in the 2+1d: `hydrodynamic' gauge fields $p_\mu, \tilde{q}_\mu$ expressing conserved bulk currents, e.g. $\mathcal{J}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu \tilde{q}_\rho, \quad \partial_\mu \mathcal{J}^\mu = 0,$

$$S_{BF}[p,\tilde{q},A,\tilde{A}] = \int_{\mathcal{M}_3} p d\tilde{q} + p d\tilde{A} + \tilde{q} dA, \qquad \mathcal{J} = *d\tilde{q}, \quad \tilde{\mathcal{J}} = *dp$$

- equations of motion

$$dp + dA = 0 \quad \rightarrow \quad p = d\theta - A$$
$$d\tilde{q} + d\tilde{A} = 0 \quad \rightarrow \quad \tilde{q} = d\psi - \tilde{A}$$

$$S_{BF}[p,\tilde{q},A,\tilde{A}] \underset{eom}{\rightarrow} S[A,\tilde{A}] = \int_{\mathcal{M}_3} (d\psi - \tilde{A}) dA$$

- check anomaly inflow

$$\tilde{\mathcal{J}}^2 \underset{eom}{=} \varepsilon^{2\alpha\beta}\partial_\alpha \left(\partial_\beta \theta - A_\beta\right) = \partial_\alpha \tilde{J}^\alpha = -\varepsilon^{\alpha\beta}\partial_\alpha A_\beta, \qquad \alpha,\beta = 0,1$$
$$\mathcal{J}^2 \underset{eom}{=} \varepsilon^{2\alpha\beta}\partial_\alpha (\partial_\beta \psi - \tilde{A}_\beta) = \partial_\alpha J^\alpha = -\varepsilon^{\alpha\beta}\partial_\alpha \tilde{A}_\beta$$

- reduction of hydrodynamic fields to 1+1d edge

$$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}\left(\partial_\beta \theta - A_\beta\right)$$
$$J^\alpha = \varepsilon^{\alpha\beta}(\partial_\beta \psi - \tilde{A}_\beta)$$

Checking the anomaly inflow

- reduction to the 1+1d edge

$$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}\,(\partial_\beta\theta - A_\beta)$$

$$J^\alpha = \varepsilon^{\alpha\beta}(\partial_\beta\psi - \tilde{A}_\beta)$$

gauge invariant

- compare with bosonic theory

$$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}\,(\partial_\beta\theta - A_\beta)$$

$$J^\alpha = \partial_\alpha\theta - A_\alpha$$

- bulk topological theory + inflow introduces `compensating' scalar fields allowing for gauge invariance of currents $J^\alpha, \tilde{J}^\alpha$ (as in sigma-models):

- θ earlier scalar field

- ψ is actually the dual field: $\partial^\mu\theta - A^\mu = \varepsilon^{\mu\nu}(\partial_\nu\psi - \tilde{A}_\nu)$

Summing up so far

- bosonic theory reproduces 1+1d chiral anomalies

- confirmed by anomaly inflow from topological theory (QHE setting, reservoir, etc), which actually suggests the need for 1+1d `hydrodynamic' fields $\theta,\ \psi$

Variational principle for perfect fluid

- Long hystory (Lichnerowitz, Carter, Arnold, Marsden, Holm...)
- recently made explicit and simple, including some anomalies (Abanov, Wiegmann, '22)
- can describes both relativistic and non-relativistic fluids

$$S[p] = \int d^n x \; P(\mu), \qquad \mu = \mu(\rho) = \mu(p_\alpha)$$

μ - chemical potential
ρ - fluid density
p_α - fluid momentum

- Euler hydrodynamics is a constrained system

- restricted variations: $\delta S[p] = 0$ for $\delta_\epsilon p_\nu = \epsilon^\alpha \partial_\alpha p_\nu + p_\alpha \partial_\nu \epsilon^\alpha, \qquad \delta_\epsilon p = \mathcal{L}_\epsilon p$

- Equations of motion take the Lichnerowicz-Carter form

$$\hat{\mathcal{J}}^\nu (\partial_\nu p_\mu - \partial_\mu p_\nu) = 0,$$ `particle current' $$\hat{\mathcal{J}}_\nu = -\frac{\delta S}{\delta p_\nu}, \quad \partial_\nu \hat{\mathcal{J}}^\nu = 0$$

in form notation $i_{\hat{\mathcal{J}}} dp = 0$

- solution in 1+1d: $dp = 0 \quad \to \quad p = d\theta$

- description equivalent to earlier bosonic theory

- can reproduce 1+1d chiral anomalies

Hydrodynamics with anomalies in 3+1d

$$S[p] = \int d^4x\, P(p)$$

- Equation of motion $\quad i_{\hat{\mathcal{J}}} dp = 0 \quad \to \quad i_{\hat{\mathcal{J}}} dpdp = 0$
- solution in 3+1d $\quad dpdp = 0, \quad \boxed{\tilde{J}^\mu = \varepsilon^{\mu\nu\rho\sigma} p_\nu \partial_\rho p_\sigma, \qquad \partial_\mu \tilde{J}^\mu = 0}$
- $\tilde{J}$ helicity current:

$$\tilde{Q} = \int d^3x\, \tilde{J}^0 \sim \int \vec{v}\cdot\vec{\omega}, \qquad \vec{\omega} = \nabla \times \vec{v}$$

- idea: identify it as axial current; couple it to backgrounds (Abanov, Wiegmann '22)

$$S[\pi, A, \tilde{A}] = \int d^4x P(\pi - A) + \tilde{A}(\pi - A) d(\pi + A), \qquad p = (\pi - A) \ \ \text{g.i.}$$

- obtain (some of) the anomalies $\quad (dpdp = 0 \ \to d\pi d\pi = 0)$

$$\partial_\mu \tilde{J}^\mu = *d[(p - A)d(\pi + A)] = - * dAdA \qquad \partial_\mu \tilde{J}^\mu = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \qquad (e/2\pi = 1)$$

$$\partial_\mu J^\mu = -2 * dAd\tilde{A}, \qquad \partial_\mu J^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} \tilde{F}_{\rho\sigma},$$

$$\partial_\nu T^\nu_\mu = F_{\mu\nu} J^\nu + \tilde{F}_{\mu\nu} \tilde{J}^\mu$$

→ anomaly is `geometric', i.e. independent of specific dynamics $P(p_\alpha)$

→ hydrodynamics can describe interacting fermions in fluid phases

Anomaly inflow from 4+1d

- 4+1d topological theory: hydrodynamic gauge fields dual to bulk currents

$$S[c,\tilde{c},p,\tilde{q},A,\tilde{A}] = \int_{\mathcal{M}_5} \tilde{c}(dA+p) + cd(\tilde{q}+\tilde{A}) + \tilde{q}dpdp + \alpha\tilde{q}d\tilde{q}d\tilde{q}, \qquad \mathcal{J} = *d\tilde{c},\ \tilde{\mathcal{J}} = *dc, \quad (\alpha = 1/3)$$

- equations of motion

$$\tilde{q} = d\psi - \tilde{A}, \qquad \tilde{c} = -2pd\tilde{q} + d\tilde{b}$$

$$p = d\theta - A, \qquad c = -pdp - 3\alpha\tilde{q}d\tilde{q} + db$$

$$S[c,\tilde{c},p,\tilde{q},A,\tilde{A}] \underset{eom}{\rightarrow} S[A,\tilde{A}] = \int_{\mathcal{M}_5} (d\psi - \tilde{A})(dAdA + \alpha d\tilde{A}d\tilde{A})$$

- obtain 3+1d anomalies by inflow

$$\partial_\mu J^\mu = -2 * dAd\tilde{A}$$

$$\partial_\mu \tilde{J}^\mu = - * dAdA - 3\alpha * d\tilde{A}d\tilde{A}$$

- compare with hydrodynamic formulation:

➡ identify fluid momentum $p = (d\theta - A) \quad \rightarrow \quad p = (\pi - A)$

➡ additional anomalous term for $\alpha \neq 0$; needs pseudoscalar field ψ

3+1d Hydrodynamics from inflow

- Idea: modify hydro action by including terms given by inflow

$$S[\pi,\psi,A,\tilde{A}] = \int_{\mathcal{M}_4} P(\pi - A) + \tilde{A}(\pi - A)d(\pi + A) + \psi(d\pi d\pi + \alpha d\tilde{A}d\tilde{A}) - \int_{\mathcal{M}_5} \tilde{A}dAdA + \alpha\tilde{A}d\tilde{A}d\tilde{A}$$

→ bulk + boundary system is gauge invariant

$p = (\pi - A), \quad \tilde{q} = (d\psi - \tilde{A})$ g.i.

- Results:

- ψ is Lagrange multiplier enforcing C-L equation, can do free variations of S
- \`geometric' bosonic theory for 3+1d interacting fermions with anomalies
- ψ additional d.o.f. of the fluid (pion-, axion-like); can also add dynamics to it without affecting anomalies
- can study theory in a chiral background $A_- = 0, \quad A_\pm = A \pm \tilde{A}$
- can also describe mixed axial-gravitational anomaly

Mixed axial-gravitational anomaly

$$D_\mu \tilde{J}^\mu = -*dAdA - 3\alpha * d\tilde{A}d\tilde{A} - \beta * \mathrm{Tr}(R^2) \qquad R_{ab} = \frac{1}{2} R_{\mu\nu,ab} dx^\mu dx^\nu, \quad \beta = 1/48$$

- No truly gravitation anomaly in 3+1d, no violation of stress-tensor conservation
- additional term to 4+1d action known from anomaly literature (index theorem)

$$\Delta S \underset{eom}{=} \beta \int_{\mathcal{M}_5} (d\psi - \tilde{A}) \mathrm{Tr}(R^2)$$

- no extra hydro fields needed
- results: $D_\mu J^\mu = -2 * dAd\tilde{A}$

$$D_\nu T^\nu_\mu = F_{\mu\nu} J^\nu + \tilde{F}_{\mu\nu} \tilde{J}^\mu + \mathrm{Tr}(R_{\mu\nu} \Sigma^\nu),$$

$\Sigma^{\mu,ab}$ spin current

- spin current is related to axial current in gravity backgrounds with torsion

- free Dirac $\qquad \Sigma^{\mu,ab} = \frac{1}{4} \bar{\Psi} \{\gamma^\mu, \sigma^{ab}\} \Psi = \frac{1}{2} \varepsilon^{\mu\nu,ab} \tilde{J}_\nu$

- ψ (axial gauge transf.) can be related to spinor rotation (local-Lorentz symm)

Conclusions

- Topological phases of matter: topological bulk and massless boundary

→ anomaly inflow: new view on anomalies

- anomaly inflow can be implemented in effective bosonic theories and hydrodynamics with Lagrangian formulation:

→ complete description of perfect fluids with anomalies in 1+1d and 3+1d

→ clearly see that anomalies parametrize universal, `geometric' effects/responses

Perspectives/extensions

- add temperature and entropy
- purely chiral fluids (Weyl fermions) in 3+1d
- 2+1d hydrodynamics (global anomaly)
- many species, non-Abelian symmetries and anomalies
- applications: topological phases, interacting fermions (eg. heavy ion coll), cosmo,….